

# A GENERAL CONSTRUCTION OF $n$ -ANGULATED CATEGORIES USING PERIODIC INJECTIVE RESOLUTIONS

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**ABSTRACT.** Let  $\mathcal{C}$  be an additive category equipped with an automorphism  $\Sigma$ . We show how to obtain  $n$ -angulations of  $(\mathcal{C}, \Sigma)$  using some particular periodic injective resolutions. We give necessary and sufficient conditions on  $(\mathcal{C}, \Sigma)$  admitting an  $n$ -angulation. Then we apply these characterizations to explain the standard construction of  $n$ -angulated categories and the  $n$ -angulated categories arising from some local rings. Moreover, we obtain a class of new examples of  $n$ -angulated categories from quasi-periodic selfinjective algebras.

## 1. INTRODUCTION

Let  $n$  be an integer greater than or equal to three. Motivated by the development of higher cluster tilting subcategories and the higher Auslander Reiten theory [15, 16], Geiss, Keller and Oppermann introduced the notion of  $n$ -angulated categories, which are “higher dimensional” analogues of triangulated categories, and gave a “standard construction” of  $n$ -angulated categories from  $(n-2)$ -cluster tilting subcategories of triangulated categories which are closed under the  $(n-2)$ -nd power of the suspension functor [11]. For  $n = 3$ , an  $n$ -angulated category is nothing but a classical triangulated category. Other examples of  $n$ -angulated categories arising from local rings were given in [6]. The theory of  $n$ -angulated categories has been developed further, see [2, 5, 7, 17, 18, 19, 20, 21, 22] for example.

The first aim and motivation of this paper is to present a general framework to unify the two constructions of  $n$ -angulated categories in [11] and [6]. The second motivation is to construct new examples of  $n$ -angulated categories.

Roughly speaking, an  $n$ -angulated category is an additive category  $\mathcal{C}$  equipped with an automorphism  $\Sigma$  of  $\mathcal{C}$  and a class  $\Theta$  of  $n$ - $\Sigma$ -sequences satisfying four axioms, denoted (N1)-(N4) (see 2.1 for precise definition). In this case,  $\Theta$  is called an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ . If  $\Theta$  satisfies (N1)-(N3), then  $\Theta$  is called a *pre- $n$ -angulation* and its elements are called  *$n$ -angles*.

We first note that if  $\mathcal{C}$  is an  $n$ -angulated category, then  $\text{mod } \mathcal{C}$ , the category of finitely presented functors from  $\mathcal{C}^{op}$  to the category  $\text{Ab}$  of abelian groups, is a Frobenius category [11, Proposition 2.5(b)]. An important property of  $n$ -angles is that all  $n$ -angles are exact [11, Proposition 2.5(a)]. By Yoneda embedding  $\iota : \mathcal{C} \rightarrow \text{proj } \mathcal{C}$ , where  $\text{proj } \mathcal{C}$  is the subcategory of  $\text{mod } \mathcal{C}$  consists of projective objects, we identify  $\mathcal{C}$  with  $\iota(\mathcal{C})$ , the essential image of  $\iota$ . Thus, an  $n$ -angle can be seen as an  $n$ - $\Sigma$ -periodic exact complex over  $\iota(\mathcal{C})$ . Furthermore, an  $n$ -angulation  $\Theta$  can be identified as a full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ , the category of  $n$ - $\Sigma$ -periodic exact

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complexes over  $\iota(\mathcal{C})$ . According to this point of view, an  $n$ -angulation turns out to be closed under translation functor and  $n$ - $\Sigma$ -homotopy equivalence. Moreover, an  $n$ -angulation must contain all  $n$ - $\Sigma$ -contractible complexes in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ .

Now assume that  $\mathcal{C}$  is an additive category equipped with an automorphism  $\Sigma$  and  $\text{mod } \mathcal{C}$  is a Frobenius category. Let  $\Theta$  be a full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  which is closed under translation functor and  $n$ - $\Sigma$ -homotopy equivalence. We construct a functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  by sending a complex  $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$  to  $M = \ker f_1$ . In this case,  $X_\bullet$  is an  $n$ - $\Sigma$ -periodic injective resolution of  $M$ . Our main theorem is the following.

**Theorem 1.1.** (cf. Theorem 3.9) *The following statements are equivalent.*

- (a) *The class  $\Theta$  is an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ .*
- (b) *The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense and  $\Theta$  satisfies axiom (N4).*
- (c) *The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense and “strongly” full.*

By definition, the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is called “strongly” full if for each morphism  $h : Z_1 X_\bullet \rightarrow Z_1 Y_\bullet$  in  $\text{mod } \mathcal{C}$ , where  $X_\bullet, Y_\bullet \in \Theta$ , there exists a morphism  $T(h) : X_\bullet \rightarrow Y_\bullet$  in  $\Theta$  such that  $Z_1(T(h)) = h$  and the mapping cone  $C(T(h))$  belongs to  $\Theta$ .

Actually, (a)  $\Leftrightarrow$  (b) and (b)  $\Rightarrow$  (c) come from a more general result, characterizing a pre- $n$ -angulation in terms of the functor  $Z_1$ .

**Theorem 1.2.** (cf. Theorem 3.8) *The following statements are equivalent.*

- (a) *The class  $\Theta$  is a pre- $n$ -angulation of  $(\mathcal{C}, \Sigma)$ .*
- (b) *The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense and full.*

Therefore, we can construct an  $n$ -angulation as follows. For each  $M \in \text{mod } \mathcal{C}$ , we fix an  $n$ - $\Sigma$ -periodic injective resolution  $T_M \in C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Denote by  $\Theta$  the full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  consisting of objects  $X_\bullet$  which is  $n$ - $\Sigma$ -homotopy equivalent to some  $T_M$ . Actually, we can fix a minimal  $n$ - $\Sigma$ -periodic injective resolution  $T_M \in C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  for each  $M \in \text{mod } \mathcal{C}$ . Then  $\Theta$  is the full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  consisting of objects isomorphic to  $T_M \oplus A_\bullet$  where  $A_\bullet$  is an  $n$ - $\Sigma$ -contractible complex. If  $\Theta$  is not closed under the translation functor, then  $\Theta$  is not an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ . Otherwise,  $\Theta$  is an  $n$ -angulation if  $\Theta$  moreover satisfies (N4). Furthermore, each  $n$ -angulation can be constructed in this way.

Recall that given a field  $k$ , a finite-dimensional  $k$ -algebra  $A$  is said to be *quasi-periodic* if  $A$  has a quasi-periodic projective resolution over the enveloping algebra  $A^e = A^{\text{op}} \otimes_k A$ , i.e.,  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$  as  $A$ - $A$ -bimodule for some natural number  $n$  and some automorphism  $\sigma$  of  $A$ . In particular,  $A$  is *periodic* if  $\Omega_{A^e}^n(A) \cong A$ . Typical examples of periodic algebras are preprojective algebras of Dynkin type, Brauer tree algebras, algebras of quaternion type, selfinjective algebra of finite representation type, some  $d$ -cluster tilted algebras, deformed mesh algebras of generalized Dynkin type and so on; see [4], [8], [9], [10].

It is well known that periodic algebras are selfinjective and their module categories are periodic, i.e., each finite dimensional module has a periodic injective resolution. More generally, assume that  $A$  is a finite-dimensional indecomposable  $k$ -algebra such that  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$  as an  $A$ - $A$ -bimodule for some automorphism  $\sigma$  of  $A$  and for some  $n \geq 3$ . Then  $A$  is a selfinjective algebra and the functor  $-\otimes_A ({}_1A_{\sigma^{-1}}) : \text{proj } A \rightarrow \text{proj } A$  is an automorphism. As an application of Theorem 1.1, we can construct a new class of examples of  $n$ -angulated categories from quasi-periodic selfinjective algebras.

**Theorem 1.3.** (*cf. Theorem 5.5*) *The category  $(\text{proj } A, - \otimes_A ({}_1 A_{\sigma^{-1}}))$  admits an  $n$ -angulation.*

This paper is organized as follows. In Section 2, we recall the definition of  $n$ -angulated category and prove some lemmas.

In Section 3, we show how to obtain  $n$ -angulations of  $(\mathcal{C}, \Sigma)$  using  $n$ - $\Sigma$ -periodic injective resolutions. We first develop some needed properties on  $n$ - $\Sigma$ -periodic injective resolution to provide some new views on  $n$ -angles. Then we prove Theorem 3.8 and Theorem 3.9. As applications, we get Corollary 3.10 and Corollary 3.11, which is a higher version of a result of Amiot [1, Theorem 8.1].

The last two sections are devoted to demonstrate that our approach is general, both for known examples and new examples.

In Section 4, we give a new point of view on known examples including algebraic triangulated categories, the standard construction of  $n$ -angulated categories given in [11] and the  $n$ -angulated categories given in [6]. Therefore, our ideas indeed present a general framework to unify the known constructions of  $n$ -angulated categories.

In Section 5, we provide some new examples of  $n$ -angulated categories. We first discuss the  $n$ -angulated structure on semisimple categories and give some new characterizations of semisimple categories; see Theorem 5.3 and Corollary 5.4. Then we apply Corollary 3.11 to obtain a large class of new examples of  $n$ -angulated categories from quasi-periodic selfinjective algebras; see Theorem 5.5. To close this section, we construct other  $n$ -angulations from known ones. We show that the group of global automorphisms of  $(\mathcal{C}, \Sigma)$  acts on the set of  $n$ -angulations of  $(\mathcal{C}, \Sigma)$  from the right; see Proposition 5.9.

## 2. DEFINITIONS AND PRELIMINARIES

Throughout this paper, we assume that  $\mathcal{C}$  is an additive category equipped with an automorphism  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ . We denote by  $\mathcal{C}(X, Y)$  the set of morphisms from  $X$  to  $Y$  in  $\mathcal{C}$ . We denote the composition of  $f \in \mathcal{C}(X, Y)$  and  $g \in \mathcal{C}(Y, Z)$  by  $gf \in \mathcal{C}(X, Z)$ . We denote by  $\text{mod } \mathcal{C}$  the category of finitely presented functors from  $\mathcal{C}^{op}$  to the category  $\text{Ab}$  of abelian groups and by  $\text{proj } \mathcal{C}$  the subcategory of  $\text{mod } \mathcal{C}$  which consists of projective objects. Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , we denote by  $F(\mathcal{C})$  the essential image of  $F$ .

An  $n$ - $\Sigma$ -sequence in  $\mathcal{C}$  is a sequence of morphisms

$$X_{\bullet} = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1).$$

Its *left rotation* is the  $n$ - $\Sigma$ -sequence

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2.$$

We can define *right rotation* of an  $n$ - $\Sigma$ -sequence similarly. An  $n$ - $\Sigma$ -sequence  $X_{\bullet}$  is *exact* if the induced sequence

$$\cdots \rightarrow \mathcal{C}(-, X_1) \rightarrow \mathcal{C}(-, X_2) \rightarrow \cdots \rightarrow \mathcal{C}(-, X_n) \rightarrow \mathcal{C}(-, \Sigma X_1) \rightarrow \cdots$$

is exact. A *trivial*  $n$ - $\Sigma$ -sequence is a sequence of the form

$$X \xrightarrow{1} X \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X$$

or any of its rotations. A *morphism* of  $n$ - $\Sigma$ -sequences is a sequence of morphisms  $\varphi_\bullet = (\varphi_1, \varphi_2, \dots, \varphi_n)$  such that the following diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_n \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1 \end{array}$$

commutes where each row is an  $n$ - $\Sigma$ -sequence. It is an *isomorphism* if  $\varphi_1, \varphi_2, \dots, \varphi_n$  are all isomorphisms in  $\mathcal{C}$ .

**Definition 2.1.** ([11]) Let  $\mathcal{C}$  be an additive category and  $\Sigma$  an automorphism of  $\mathcal{C}$ . A collection  $\Theta$  of  $n$ - $\Sigma$ -sequences is called a *pre- $n$ -angulation* and its elements are called  *$n$ -angles* if  $\Theta$  satisfies the following axioms:

(N1) (a) The class  $\Theta$  is closed under isomorphisms, direct sums and direct summands.

(b) For each object  $X \in \mathcal{C}$  the trivial sequence

$$X \xrightarrow{1} X \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X$$

belongs to  $\Theta$ .

(c) For each morphism  $f_1 : X_1 \rightarrow X_2$  in  $\mathcal{C}$ , there exists an  $n$ - $\Sigma$ -sequence in  $\Theta$  whose first morphism is  $f_1$ .

(N2) An  $n$ - $\Sigma$ -sequence belongs to  $\Theta$  if and only if its left rotation belongs to  $\Theta$ .

(N3) Each commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_n \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1 \end{array}$$

with rows in  $\Theta$  can be completed to a morphism of  $n$ - $\Sigma$ -sequences.

In this case, we say  $(\mathcal{C}, \Sigma, \Theta)$  is a *pre- $n$ -angulated category*. If  $\Theta$  moreover satisfies the following axiom, then it is called an  *$n$ -angulation* of  $(\mathcal{C}, \Sigma)$  and  $(\mathcal{C}, \Sigma, \Theta)$  is called an  *$n$ -angulated category*:

(N4) In the situation of (N3), the morphisms  $\varphi_3, \dots, \varphi_n$  can be chosen such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ \Sigma \varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to  $\Theta$ .

In the rest of this section, we will collect some facts on pre- $n$ -angulated categories.

**Lemma 2.2.** ([11, Proposition 2.5]) *Let  $(\mathcal{C}, \Sigma, \Theta)$  be a pre- $n$ -angulated category. Then the following hold.*

(a) *All  $n$ -angles in  $\mathcal{C}$  are exact.*

(b) *The category  $\text{mod } \mathcal{C}$  is an abelian Frobenius category.*

**Lemma 2.3.** *Let  $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$  be an  $n$ -angle. Then the following statements are equivalent.*

(a)  *$f_1$  is a split monomorphism;*

- (b)  $f_{n-1}$  is a split epimorphism;  
(c)  $f_n = 0$ .

*Proof.* We only prove (a)  $\Leftrightarrow$  (c), since we can prove (b)  $\Leftrightarrow$  (c) similarly. If  $f_1$  is a split monomorphism, then there exists a morphism  $f'_1 : X_2 \rightarrow X_1$  such that  $f'_1 f_1 = 1$ . By (N3), we obtain the following commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \parallel & & \downarrow f'_1 & & \downarrow & & & & \downarrow & & \parallel \\ X_1 & \xrightarrow{1} & X_1 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \Sigma X_1 \end{array}$$

whose rows are  $n$ -angles. Thus  $f_n = 0$ . Conversely, if  $f_n = 0$ , then the diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \parallel & & & & & & & & \downarrow & & \parallel \\ X_1 & \xrightarrow{1} & X_1 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \Sigma X_1 \end{array}$$

can be completed as a morphism of  $n$ -angles by (N2) and (N3), which implies that  $f_1$  is a split monomorphism.  $\square$

**Lemma 2.4.** *Let  $f : X \rightarrow Y$  be a morphism in a pre- $n$ -angulated category  $\mathcal{C}$ .*

- (a) *If  $f$  is a monomorphism, then  $f$  is a split monomorphism;*  
(b) *If  $f$  is an epimorphism, then  $f$  is a split epimorphism.*

*Proof.* (a) By (N1)(c), we assume that  $X \xrightarrow{f} Y \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X$  is an  $n$ -angle. Since  $f \cdot \Sigma^{-1} f_n = 0$  and  $f$  is a monomorphism, we have  $\Sigma^{-1} f_n = 0$ . Thus  $f_n = 0$ . By Lemma 2.3,  $f$  is a split monomorphism. We can prove (b) dually.  $\square$

### 3. PERIODIC INJECTIVE RESOLUTIONS AND $n$ -ANGULATIONS

In this section, we always assume that  $\text{mod } \mathcal{C}$  is a Frobenius category, which may be implied in other condition; see Lemma 3.3(a). Thus the stable category  $\underline{\text{mod}} \mathcal{C}$  is a triangulated category with the suspension functor  $\Omega^{-1}$ . The automorphism  $\Sigma$  of  $\mathcal{C}$  induces an automorphism  $\Sigma$  of  $\text{mod } \mathcal{C}$  by mapping  $M$  to  $M \cdot \Sigma^{-1}$ . The Yoneda functor  $\iota : \mathcal{C} \rightarrow \text{proj } \mathcal{C}$ , which maps  $A$  to  $\mathcal{C}(-, A)$ , gives a natural equivalence between  $\mathcal{C}$  and  $\iota(\mathcal{C})$ . We note that  $\iota(\mathcal{C})$  is not equal to  $\text{proj } \mathcal{C}$  in general unless  $\mathcal{C}$  is idempotent complete. For convenience, we identify  $\mathcal{C}$  with  $\iota(\mathcal{C})$ .

**3.1.  $n$ - $\Sigma$ -periodic injective resolutions.** An exact complex  $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$  over  $\iota(\mathcal{C})$  is called  $n$ - $\Sigma$ -periodic if  $X_{k+n} = \Sigma X_k$  and  $f_{k+n} = \Sigma f_k$  for all  $k \in \mathbb{Z}$ . Since we have identified  $\mathcal{C}$  with  $\iota(\mathcal{C})$ , we can identify an exact  $n$ - $\Sigma$ -sequence in  $\mathcal{C}$  as an  $n$ - $\Sigma$ -periodic exact complex over  $\iota(\mathcal{C})$ , and vice versa.

A morphism  $\varphi_\bullet = (\varphi_i)_{i \in \mathbb{Z}}$  between two  $n$ - $\Sigma$ -periodic exact complexes  $X_\bullet$  and  $Y_\bullet$  is given by a morphism of complexes such that  $\varphi_{k+n} = \Sigma \varphi_k$  for all  $k \in \mathbb{Z}$ . We denote by  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  the category of  $n$ - $\Sigma$ -periodic exact complexes over  $\iota(\mathcal{C})$ . Let  $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}, Y_\bullet = (Y_i, g_i)_{i \in \mathbb{Z}}$  be two objects in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ , and  $\varphi_\bullet, \psi_\bullet$  be two morphisms from  $X_\bullet$  to  $Y_\bullet$ . An  $n$ - $\Sigma$ -homotopy from  $\varphi_\bullet$  to  $\psi_\bullet$  is given by morphisms  $h_i : X_{i+1} \rightarrow Y_i$  such that  $\varphi_i - \psi_i = h_i f_i + g_{i-1} h_{i-1}$  and  $h_{n+i} = \Sigma h_i$  for all  $i \in \mathbb{Z}$ . In this case, we say that  $\varphi_\bullet$  and  $\psi_\bullet$  are  $n$ - $\Sigma$ -homotopic. Since  $n$ - $\Sigma$ -homotopy relation

is an equivalence relation, we can form the relative homotopy category  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ , by considering the objects are the same as those of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  and the additive group of  $n$ - $\Sigma$ -homotopy classes of morphisms from  $X_\bullet$  to  $Y_\bullet$  in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  as the group of morphisms from  $X_\bullet$  to  $Y_\bullet$  in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ .

**Lemma 3.1.** *The relative homotopy category  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  is a triangulated category.*

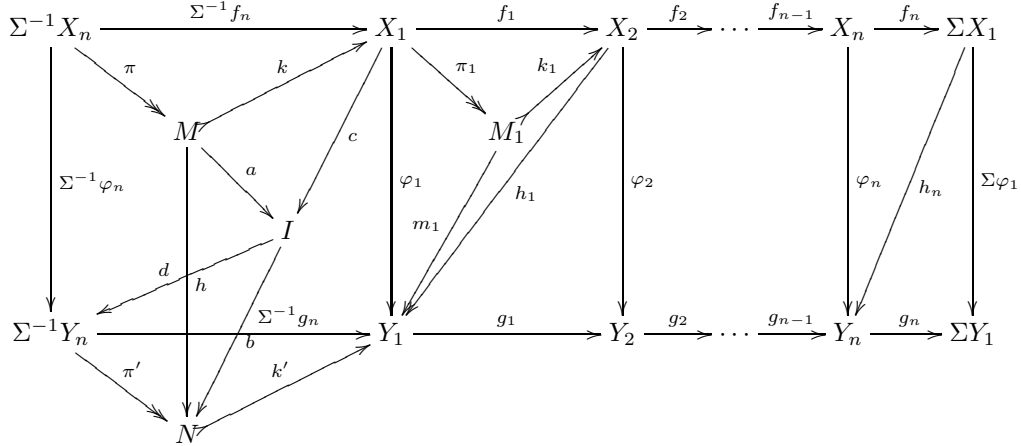
*Proof.* Denote by  $\mathcal{S}$  the class of all chain-wise split exact sequences  $0 \rightarrow X_\bullet \xrightarrow{\varphi_\bullet} Y_\bullet \xrightarrow{\psi_\bullet} Z_\bullet \rightarrow 0$  in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Then  $(C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C})), \mathcal{S})$  is an exact category. Similar to [23, Proposition 7.1], we can show that  $(C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C})), \mathcal{S})$  is a Frobenius category and the projective-injectives are the  $n$ - $\Sigma$ -contractible complexes, whose identity morphisms are  $n$ - $\Sigma$ -homotopic to the zero morphisms. Thus the stable category  $\underline{C}_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C})) = K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  is a triangulated category whose suspension functor is the translation functor.  $\square$

**Lemma 3.2.** *The following hold.*

- (a) *The functor  $Z_1 : C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C})) \rightarrow \text{mod}\mathcal{C}$ , which sends a complex  $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$  to  $\ker f_1$ , induces a triangle functor  $\underline{Z}_1 : K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C})) \rightarrow \underline{\text{mod}}\mathcal{C}$ .*
- (b) *If  $\underline{Z}_1(\varphi_\bullet) = 0$ , then  $\varphi_\bullet$  is  $n$ - $\Sigma$ -homotopic to some morphism  $\varphi'_\bullet = (0, 0, \dots, 0, \varphi'_n)$ .*
- (c) *The kernel of  $\underline{Z}_1$  is an ideal whose square vanishes.*

*Proof.* (a) By the Snake Lemma, we can show that  $Z_1$  is an exact functor. Then  $Z_1$  induces a triangle functor  $\underline{Z}_1$  since  $Z_1$  preserves the projective-injectives.

(b) Let  $k : M \rightarrow X_1$  be the kernel of  $f_1$  and  $k' : N \rightarrow Y_1$  be the kernel of  $g_1$ . We assume that  $\Sigma^{-1}f_n = k\pi$  and  $\Sigma^{-1}g_n = k'\pi'$ . Since  $\underline{Z}_1(\varphi_\bullet) = 0$ , the morphism  $h = Z_1(\varphi_\bullet)$  admits a factorization  $M \xrightarrow{a} I \xrightarrow{b} N$ , where  $I$  is projective-injective. Thus there exist two morphisms  $c : X_1 \rightarrow I$  and  $d : I \rightarrow \Sigma^{-1}Y_n$  such that  $a = ck$  and  $b = \pi'd$ . We put  $h'_n = \Sigma(dc)$ . Now we assume that  $f_i : X_i \rightarrow X_{i+1}$  has a factorization  $k_i\pi_i : X_i \twoheadrightarrow M_i \rightarrow X_{i+1}$  for  $i = 1, 2, \dots, n-1$ . Note that  $(\varphi_1 - \Sigma^{-1}(g_n h'_n))\Sigma^{-1}f_n = 0$ , hence there exists a morphism  $m_1 : M_1 \rightarrow Y_1$  such that  $\varphi_1 - \Sigma^{-1}(g_n h'_n) = m_1\pi_1$ . Since  $Y_1$  is projective-injective, there exists a morphism  $h_1 : X_2 \rightarrow Y_1$  such that  $m_1 = h_1\pi_1$ . Thus  $\varphi_1 = \Sigma^{-1}(g_n h'_n) + h_1\pi_1$ . Similarly there exist morphisms  $h_i : X_{i+1} \rightarrow Y_i$  for  $2 \leq i \leq n$  such that  $\varphi_i = h_i\pi_i + g_{i-1}h_{i-1}$ . Suppose that  $\varphi'_n = (h_n - h'_n)f_n$ , then  $\varphi_n - \varphi'_n = g_{n-1}h_{n-1} + h'_nf_n$ .



Hence the morphism  $\varphi_\bullet$  is  $n$ - $\Sigma$ -homotopic to the morphism  $\varphi'_\bullet = (0, 0, \dots, 0, \varphi'_n)$  with an  $n$ - $\Sigma$ -homotopy  $(h_1, h_2, \dots, h_{n-1}, h'_n)$ .

(c) Let  $\varphi_\bullet : X_\bullet \rightarrow Y_\bullet$  and  $\psi_\bullet : Y_\bullet \rightarrow Z_\bullet$  be two morphisms in the kernel of  $\underline{Z}_1$ . Up to  $n$ - $\Sigma$ -homotopy, we assume that  $\varphi_\bullet = (0, 0, \dots, 0, \varphi_n)$  and  $\psi_\bullet = (0, 0, \dots, 0, \psi_n)$ . Thus we get the following commutative diagram

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-2} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
 \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 & \nearrow a_n & \downarrow \varphi_n & & \downarrow 0 \\
 Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \\
 \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 & & \downarrow \psi_n & \nearrow b_{n+1} & \downarrow 0 \\
 Z_1 & \xrightarrow{h_1} & Z_2 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{n-2}} & Z_{n-1} & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma Z_1
 \end{array}$$

Since  $g_n \varphi_n = 0$  and  $\psi_n g_{n-1} = 0$ ,  $\varphi_n$  factors through  $g_{n-1}$  and  $\psi_n$  factors through  $g_n$ . Thus  $\psi_n \varphi_n = b_{n+1} g_n g_{n-1} a_n = 0$ . Therefore,  $\psi_\bullet \varphi_\bullet = 0$ .  $\square$

**Lemma 3.3.** *Let  $\Theta$  be a full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\mathcal{C})$  and  $\varphi_\bullet : X_\bullet \rightarrow Y_\bullet$  be a morphism in  $\Theta$ .*

(a) *If the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is full, then  $\text{mod } \mathcal{C}$  is Frobenius.*

(b) *If the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is full, then  $Z_1(\varphi_\bullet)$  is an isomorphism in  $\underline{\text{mod}} \mathcal{C}$  if and only if  $\varphi_\bullet$  is an  $n$ - $\Sigma$ -homotopy equivalence.*

(c) *If the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is full and dense, then for each  $M \in \text{mod } \mathcal{C}$ , up to  $n$ - $\Sigma$ -homotopy equivalence, there exists a unique  $T_M \in \Theta$  such that  $Z_1 T_M \cong M$ .*

*Proof.* (a) This is an adaptation of the proof of [11, Proposition 2.5(b)]. It is clear that  $\text{mod } \mathcal{C}$  has enough projectives. We claim that each projective  $\mathcal{C}(-, A)$  is an injective. Indeed, for each  $M \in \text{mod } \mathcal{C}$ , we note that  $\Sigma^{-1} M \in \text{mod } \mathcal{C}$ . Since  $Z_1$  is dense, there exists an object  $X_\bullet \in \Theta$  such that  $Z_1 X_\bullet \cong \Sigma^{-1} M$ . Thus  $X_\bullet[-1]$  is an  $n$ - $\Sigma$ -periodic projective resolution of  $M$ . For each  $i \geq 1$ , by the Yoneda Lemma, it is not hard to see that  $\text{Ext}^i(M, \mathcal{C}(-, A)) \cong H^i(X_\bullet[-1], \mathcal{C}(-, A)) = 0$  since  $X_\bullet[-1]$  is exact. Consequently, each projective is an injective. Since  $Z_1$  is dense, it is easy to see that  $\text{mod } \mathcal{C}$  has enough injectives, moreover, the projectives and injectives coincide. Therefore,  $\text{mod } \mathcal{C}$  is Frobenius.

(b) The “if” part is obvious. For the “only if” part, we only need to show that the functor  $\underline{Z}_1 : \Theta \rightarrow \underline{\text{mod}} \mathcal{C}$  detects isomorphisms. Since a full functor whose kernel is an ideal such that square vanishes detects isomorphisms [1, Lemma 8.6], the “only if” part follows from Lemma 3.2(c).

(c) It is a direct consequence of (b).  $\square$

**Remark 3.4.** (a) Let  $\Theta_1 \subseteq \Theta_2$  be full subcategories of  $C_{n-\Sigma}^{\text{ex}}(\mathcal{C})$  which are closed under  $n$ - $\Sigma$ -homotopy equivalence. If the functors  $Z_1 : \Theta_1 \rightarrow \text{mod } \mathcal{C}$  and  $Z_1 : \Theta_2 \rightarrow \text{mod } \mathcal{C}$  are full and dense, then  $\Theta_1 = \Theta_2$ .

(b) Assume that  $X_\bullet \in C_{n-\Sigma}^{\text{ex}}(\mathcal{C})$  and  $M \in \text{mod } \mathcal{C}$ . If  $Z_1 X_\bullet \cong M$ , then  $X_\bullet$  is an  $n$ - $\Sigma$ -periodic injective resolution of  $M$ . Moreover,  $X_\bullet$  is an  $n$ - $\Sigma$ -periodic injective resolution of  $M$  if and only if  $X_\bullet[-1]$  is an  $n$ - $\Sigma$ -periodic projective resolution of  $\Sigma M$ .

**3.2. The class of pre- $n$ -angulations.** Since  $\text{mod } \mathcal{C}$  is a Frobenius category, for each  $M \in \text{mod } \mathcal{C}$ , we fix a short exact sequence  $0 \rightarrow M \rightarrow I_M \rightarrow \Omega^{-1} M \rightarrow 0$  with

$I_M \in \text{proj } \mathcal{C}$ . Thus we obtain a standard injective resolution of  $M$  as follows:

$$I_M \rightarrow I_{\Omega^{-1}M} \rightarrow I_{\Omega^{-2}M} \rightarrow \cdots \quad (3.1)$$

Note that the automorphism  $\Sigma$  of  $\text{mod } \mathcal{C}$  is an exact functor. We may assume that  $I_{\Sigma M} = \Sigma I_M$  and  $\Sigma \Omega^{-1}M = \Omega^{-1}\Sigma M$ . In fact, the automorphism  $\Sigma$  of  $\mathcal{C}$  induces a triangle functor  $(\Sigma, \sigma)$  on  $\underline{\text{mod}} \mathcal{C}$ . It is well known that  $(\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  is also a triangle functor on  $\underline{\text{mod}} \mathcal{C}$ .

Suppose that there exists an isomorphism  $\alpha : (\Sigma, \sigma) \rightarrow (\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  of triangle functors. Let  $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$  be an exact  $n$ - $\Sigma$ -sequence in  $\mathcal{C}$  and  $M = \ker f_1$ . Note that  $X_\bullet$  can be seen as an  $n$ - $\Sigma$ -periodic injective resolution of  $M$  and  $f_n$  has a factorization  $X_n \twoheadrightarrow \Sigma M \hookrightarrow \Sigma X_1$ , so by the Comparison Theorem [24, Theorem 6.16], there exists an isomorphism  $\beta_M : \Sigma M \xrightarrow{\sim} \Omega^{-n}M$  in  $\underline{\text{mod}} \mathcal{C}$ . We denote by  $\Theta_\alpha$  the class of exact  $n$ - $\Sigma$ -sequences  $X_\bullet$  in  $\mathcal{C}$  such that  $\beta_M = \alpha_M$ , where  $M = Z_1 X_\bullet$ .

**Lemma 3.5.** ([11, Proposition 3.4]) *Let  $(\mathcal{C}, \Sigma, \Theta)$  be a pre- $n$ -angulated category. Then the map from the class of isomorphisms of triangle functors between  $(\Sigma, \sigma)$  and  $(\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  to the class of pre- $n$ -angulations of  $(\mathcal{C}, \Sigma)$ , mapping  $\alpha$  to  $\Theta_\alpha$ , is a bijection.*

It follows from (N2) that a pre- $n$ -angulation is closed under translation functor. The following lemma implies that a pre- $n$ -angulation is closed under  $n$ - $\Sigma$ -homotopy equivalence.

**Lemma 3.6.** *Suppose that  $(\mathcal{C}, \Sigma, \Theta)$  is a pre- $n$ -angulated category. Let  $X_\bullet, Y_\bullet$  be two objects in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ , and  $\varphi_\bullet : X_\bullet \rightarrow Y_\bullet$  be an  $n$ - $\Sigma$ -homotopy equivalence. Then  $X_\bullet \in \Theta$  if and only if  $Y_\bullet \in \Theta$ .*

*Proof.* By Lemma 3.5, we can assume that  $\Theta = \Theta_\alpha$  for some isomorphism  $\alpha : (\Sigma, \sigma) \rightarrow (\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  of triangle functors. Assume that  $M = Z_1 X_\bullet$  and  $N = Z_1 Y_\bullet$ , then  $\varphi_\bullet$  induces a morphism  $h = Z_1(\varphi_\bullet) : M \rightarrow N$ . We note that  $h$  is an isomorphism in  $\underline{\text{mod}} \mathcal{C}$  since  $\varphi_\bullet$  is an isomorphism in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . By the Comparison Theorem we have  $\Omega^{-n}h \cdot \beta_M = \beta_N \cdot \Sigma h$  in  $\underline{\text{mod}} \mathcal{C}$ . Note that we also have  $\Omega^{-n}h \cdot \alpha_M = \alpha_N \cdot \Sigma h$  by the naturality of  $\alpha$ . Thus

$$\begin{aligned} X_\bullet \in \Theta &\Leftrightarrow \beta_M = \alpha_M \\ &\Leftrightarrow (\Omega^{-n}h)^{-1} \cdot \beta_N \cdot \Sigma h = (\Omega^{-n}h)^{-1} \cdot \alpha_N \cdot \Sigma h \\ &\Leftrightarrow \beta_N = \alpha_N \Leftrightarrow Y_\bullet \in \Theta. \end{aligned}$$

□

**Corollary 3.7.** (see [6, Lemma 2.2]) *Let  $(\mathcal{C}, \Sigma, \Theta)$  be a pre- $n$ -angulated category. Then  $\Theta$  must include all contractible  $n$ - $\Sigma$ -sequences.*

*Proof.* We note that trivial  $n$ - $\Sigma$ -sequences and contractible  $n$ - $\Sigma$ -sequences are  $n$ - $\Sigma$ -homotopy equivalent, because they are zero objects in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Since each trivial  $n$ - $\Sigma$ -sequence belongs to  $\Theta$ , by Lemma 3.6 we infer that each contractible  $n$ - $\Sigma$ -sequence belongs to  $\Theta$  too. □

Now we are ready to prove the following result.

**Theorem 3.8.** *Let  $\Theta$  be a full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  which is closed under translation functor and  $n$ - $\Sigma$ -homotopy equivalence. Then the following statements are equivalent.*



- (a) The class  $\Theta$  is a pre- $n$ -angulation of  $(\mathcal{C}, \Sigma)$ .  
 (b) The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is full and dense.  
 (c) There exists an isomorphism  $\alpha : (\Sigma, \sigma) \rightarrow (\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  of triangle functors such that  $\Theta = \Theta_\alpha$ .

*Proof.* (a)  $\Rightarrow$  (b). For each  $M \in \text{mod } \mathcal{C}$ , we fix a projective presentation

$$\mathcal{C}(-, X_1) \xrightarrow{\mathcal{C}(-, f_1)} \mathcal{C}(-, X_2) \rightarrow M \rightarrow 0.$$

By (N1)(c), we get an  $n$ -angle

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1) \in \Theta.$$

It follows from (N2) that  $T_M = X_\bullet[2] \in \Theta$  and  $Z_1 T_M = M$ .

For each morphism  $h : M \rightarrow N$  in  $\text{mod } \mathcal{C}$ , let  $T_M = (X_i, f_i)_{i \in \mathbb{Z}} \in \Theta$  and  $T_N = (Y_i, g_i)_{i \in \mathbb{Z}} \in \Theta$  such that  $Z_1 T_M = M$  and  $Z_1 T_N = N$ . Since  $Y_1$  and  $Y_2$  are projective-injectives, we obtain the following commutative diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ & & \downarrow h & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\ 0 & \longrightarrow & N & \longrightarrow & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

whose rows are exact sequences. By (N3) there exist morphisms  $\varphi_i : X_i \rightarrow Y_i$  for  $3 \leq i \leq n$ , such that the above diagram is commutative. We denote by  $T(h) = (\varphi_1, \varphi_2, \dots, \varphi_n)$ . Then  $Z_1 T(h) = h$ .

(b)  $\Rightarrow$  (c). Given an object  $M \in \text{mod } \mathcal{C}$ , assume that  $T_M = (X_i, f_i)_{i \in \mathbb{Z}} \in \Theta$  such that  $Z_1 T_M \cong M$ . By (3.1), we obtain the following commutative diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \Sigma M & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \alpha_M & & \\ 0 & \longrightarrow & M & \longrightarrow & I_M & \longrightarrow & I_{\Omega^{-1}M} & \longrightarrow & \cdots & \longrightarrow & I_{\Omega^{1-n}M} & \longrightarrow & \Omega^{-n}M & \longrightarrow & 0 \end{array}$$

with exact rows. The Comparison Theorem implies that  $\alpha_M : \Sigma M \rightarrow \Omega^{-n}M$  is an isomorphism in  $\underline{\text{mod}} \mathcal{C}$ . For each morphism  $h : M \rightarrow N$  in  $\text{mod } \mathcal{C}$ , we deduce that  $\Omega^{-n}h \cdot \alpha_M = \alpha_N \cdot \Sigma h$  in  $\underline{\text{mod}} \mathcal{C}$  by the Comparison Theorem. Thus we obtain a functorial isomorphism  $\alpha : \Sigma \rightarrow \Omega^{-n}$ . For each  $M \in \text{mod } \mathcal{C}$ , we may assume that  $X_1 = I_M$  and  $0 \rightarrow M \xrightarrow{a_1} X_1 \xrightarrow{b_1} \Omega^{-1}M \rightarrow 0$  is an exact sequence. By Lemma 3.3(b), we have  $Z_1(T_M[1]) \cong \Omega^{-1}M$ . By the construction of the functor  $\alpha$ , we obtain the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma M & \xrightarrow{\Sigma a_1} & \Sigma X_1 & \xrightarrow{(-1)^n \Sigma b_1} & \Sigma \Omega^{-1}M \longrightarrow 0 \\ & & \downarrow \alpha_M & & \downarrow & & \downarrow \alpha_{\Omega^{-1}M} \\ 0 & \longrightarrow & \Omega^{-n}M & \longrightarrow & I_{\Omega^{-n}M} & \longrightarrow & \Omega^{-n-1}M \longrightarrow 0 \end{array}$$

with exact rows. Since  $\Sigma X_1 = I_{\Sigma M}$ , we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma M & \xrightarrow{\Sigma a_1} & \Sigma X_1 & \xrightarrow{\Sigma b_1} & \Omega^{-1} \Sigma M \longrightarrow 0 \\
 & & \downarrow \alpha_M & & \downarrow & & \downarrow \Omega^{-1} \alpha_M \\
 0 & \longrightarrow & \Omega^{-n} M & \longrightarrow & I_{\Omega^{-n} M} & \longrightarrow & \Omega^{-n-1} M \longrightarrow 0
 \end{array}$$

with exact rows. Therefore,  $\alpha_{\Omega^{-1} M} = (-1)^n \Omega^{-1} \alpha_M$  in  $\text{mod } \mathcal{C}$ . In other words,  $\alpha : (\Sigma, \sigma) \rightarrow (\Omega^{-n}, (-1)^n 1_{\Omega^{-n-1}})$  is an isomorphism of triangle functors. Now  $(\mathcal{C}, \Sigma)$  admits a pre- $n$ -angulation  $\Theta_\alpha$  by Lemma 3.5.

It remains to show that  $\Theta = \Theta_\alpha$ . By the construction of  $\alpha$ , we have  $\Theta \subseteq \Theta_\alpha$ . Let  $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$  be an object in  $\Theta_\alpha$  and  $M$  be the kernel of  $f_1$ . Suppose  $T_M = (Y_i, g_i)_{i \in \mathbb{Z}} \in \Theta$  such that  $Z_1 T_M = M$ . Since  $Y_1$  and  $Y_2$  are projective-injective, we find morphisms  $\varphi_1 : X_1 \rightarrow Y_1$  and  $\varphi_2 : X_2 \rightarrow Y_2$  such that the following diagram

$$\begin{array}{ccccccccccc}
 & & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
 & \nearrow & \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\
 M & & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \Sigma Y_1 \\
 & \nwarrow & & & & & & & & & & & \\
 & & M & & & & & & & & & & 
 \end{array}$$

commutes. We complete  $(\varphi_1, \varphi_2)$  to an  $n$ - $\Sigma$ -periodic morphism  $\varphi_\bullet = (\varphi_1, \varphi_2, \dots, \varphi_n)$  from  $X_\bullet$  to  $T_M$  by (N3). Since  $Z_1(\varphi_\bullet) = 1$ , we obtain that  $\varphi_\bullet : X_\bullet \rightarrow T_M$  is an  $n$ - $\Sigma$ -homotopy equivalence by Lemma 3.3(a). Thus  $X_\bullet \in \Theta$  because  $\Theta$  is closed under  $n$ - $\Sigma$ -homotopy equivalence. Therefore,  $\Theta = \Theta_\alpha$ .

(c)  $\Rightarrow$  (a). It follows from Lemma 3.5.  $\square$

**3.3.  $n$ -angulations.** Keep the notation as above. The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is called “strongly” full if for each morphism  $h : Z_1 X_\bullet \rightarrow Z_1 Y_\bullet$  in  $\text{mod } \mathcal{C}$ , where  $X_\bullet, Y_\bullet \in \Theta$ , there exists a morphism  $T(h) : X_\bullet \rightarrow Y_\bullet$  in  $\Theta$  such that  $Z_1(T(h)) = h$  and the mapping cone  $C(T(h))$  belongs to  $\Theta$ .

The main theorem of this paper is the following.

**Theorem 3.9.** *Let  $\Theta$  be a full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  which is closed under translation functor and  $n$ - $\Sigma$ -homotopy equivalence. Then the following statements are equivalent.*

- (a) *The class  $\Theta$  is an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ .*
- (b) *The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense and  $\Theta$  satisfies axiom (N4).*
- (c) *The functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense and “strongly” full.*

*Proof.* We note that if  $\Theta$  satisfies axiom (N4), then it is easy to see that the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is full. Thus (a)  $\Leftrightarrow$  (b) and (b)  $\Rightarrow$  (c) follows from Theorem 3.8.

(c)  $\Rightarrow$  (b). Given a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in  $\Theta$ . Assume that  $k_1 : M \rightarrow X_1$  is the kernel of  $f_1$ ,  $k'_1 : N \rightarrow Y_1$  is the kernel of  $g_1$  and  $h : M \rightarrow N$  is the induced morphism. By assumption, there is an object  $T(h) = (\phi_1, \phi_2, \dots, \phi_n) \in \Theta$  such that  $Z_1 T(h) = h$  and the mapping cone  $C(T(h)) \in \Theta$ . Since  $(\varphi_1 - \phi_1)k_1 = 0$ , there exists a morphism  $h_1 : X_2 \rightarrow Y_1$  such that  $\varphi_1 - \phi_1 = h_1 f_1$ . Note that  $(\varphi_2 - \phi_2 - g_1 h_1) f_1 = 0$ . There exists a morphism  $h_2 : X_3 \rightarrow Y_2$  such that  $\varphi_2 - \phi_2 = g_1 h_1 + h_2 f_2$ . We define  $\varphi_3 = \phi_3 + g_2 h_2$ , then  $g_3 \varphi_3 = g_3 \phi_3 = \phi_4 f_3$ . If we take  $\varphi_4 = \phi_4, \dots, \varphi_n = \phi_n$ , then  $g_i \varphi_i = \varphi_{i+1} f_i$  for  $4 \leq i \leq n-1$ , and  $g_n \varphi_n = g_n \phi_n = \Sigma \phi_1 \cdot f_n = \Sigma(\varphi_1 - h_1 f_1) \cdot f_n = \Sigma \varphi_1 \cdot f_n$ . Thus  $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$  is an  $n$ - $\Sigma$ -periodic morphism and  $\varphi_\bullet$  is  $n$ - $\Sigma$ -homotopic to  $T(h)$  with the  $n$ - $\Sigma$ -homotopy  $(h_1, h_2, 0, \dots, 0)$ . Hence the mapping cone  $C(\varphi_\bullet)$  is isomorphic to the mapping cone  $C(T(h))$  in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Consequently,  $C(\varphi_\bullet) \in \Theta$ .  $\square$

**Corollary 3.10.** *If there exists a triangle functor  $T : \underline{\text{mod}}\mathcal{C} \rightarrow K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  such that  $Z_1 T = \text{Id}$ , then  $(\mathcal{C}, \Sigma, \Theta)$  is an  $n$ -angulated category, where  $\Theta = T(\underline{\text{mod}}\mathcal{C})$ .*

*Proof.* Since  $Z_1 T = \text{Id}$ , the functor  $Z_1 : \Theta \rightarrow \text{mod}\mathcal{C}$  is dense and full. For each morphism  $h : M \rightarrow N$  in  $\text{mod}\mathcal{C}$ , assume that  $M \xrightarrow{h} N \rightarrow L \rightarrow \Omega^{-1}M$  is a triangle in  $\underline{\text{mod}}\mathcal{C}$ . Then  $TM \xrightarrow{T(h)} TN \rightarrow TL \rightarrow (TM)[1]$  is a triangle in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  since  $T$  is a triangle functor. It follows that the mapping cone  $C(T(h)) \cong TL$  in  $K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Thus  $C(T(h)) \in \Theta$ . Now the corollary holds by Theorem 3.9.  $\square$

The following corollary is a higher version of [1, Theorem 8.1]. It provides a unified way to construct  $n$ - $\Sigma$ -injective resolutions. We will give some applications in Section 5.2.

**Corollary 3.11.** *Assume there exists an exact sequence of exact endofunctors of  $\text{mod}\mathcal{C}$*

$$0 \rightarrow \text{Id} \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \rightarrow X^n \rightarrow \Sigma \rightarrow 0 \quad (3.2)$$

*where all the  $X^i$  take values in  $\iota(\mathcal{C})$ . Then  $(\mathcal{C}, \Sigma)$  admits an  $n$ -angulation.*

*Proof.* For each  $M \in \text{mod}\mathcal{C}$ , we denote by  $T_M$  the following  $n$ - $\Sigma$ -periodic exact complex

$$X^1 M \rightarrow X^2 M \rightarrow \cdots \rightarrow X^n M \rightarrow \Sigma X^1 M$$

induced by the exact sequence (3.2). Then  $Z_1 T_M = M$ . Since all  $X^i$  are exact functors, it is easy to see that the functor  $T : \text{mod}\mathcal{C} \rightarrow C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ , which sends an object  $M$  to  $T_M$ , is an exact functor preserving the projective-injectives. Thus  $T$  induces a triangle functor  $T : \underline{\text{mod}}\mathcal{C} \rightarrow K_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  such that  $Z_1 T = \text{Id}$ . By Corollary 3.10, we are done.  $\square$

**Remark 3.12.** Let  $\mathcal{C}$  be an additive category and  $\Sigma$  be an automorphism of  $\mathcal{C}$ . By Theorem 3.9, we can construct an  $n$ -angulation as follows. For each  $M \in \text{mod}\mathcal{C}$ , we fix an  $n$ - $\Sigma$ -periodic injective resolution  $T_M \in C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$ . Denote by  $\Theta$  the full subcategory of  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  consisting of objects  $X_\bullet$  which is  $n$ - $\Sigma$ -homotopy

equivalent to some  $T_M$ . If  $\Theta$  is not closed under the translation functor, then  $\Theta$  is not an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ . Otherwise,  $\Theta$  is an  $n$ -angulation if  $\Theta$  moreover satisfies (N4). More details and examples will be given in Section 4 and Section 5.

#### 4. FIRST EXAMPLES

In this section, we will apply the ideas of Section 3 to unify the  $n$ -angulated structure of known examples including algebraic triangulated categories, the standard construction of  $n$ -angulated categories and the  $n$ -angulated categories from local rings. We don't plan to prove that they are  $n$ -angulated categories. Instead, we only want to understand the  $n$ -angulated structure with a new point of view.

**4.1. Algebraic triangulated categories.** Let  $(\mathcal{B}, \mathcal{S})$  be a Frobenius category. We denote by  $\underline{\mathcal{B}}$  the stable category  $\mathcal{B}/[\mathcal{I}]$ , where  $\mathcal{I}$  is the full subcategory consisting of projective-injectives. Given a morphism  $f : X \rightarrow Y$  in  $\mathcal{B}$ , we denote by  $\underline{f}$  the image of  $f$  in  $\underline{\mathcal{B}}$  under the canonical functor  $\mathcal{B} \rightarrow \underline{\mathcal{B}}$ . By [13, Proposition 2.2], there exists an automorphism  $\Sigma : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}}$ . We recall the definition of  $\Sigma$  as follows. For each  $X \in \mathcal{B}$ , we fix a short exact sequence  $0 \rightarrow X \xrightarrow{i_X} I(X) \xrightarrow{p_X} \Sigma X \rightarrow 0$  in  $\mathcal{S}$  such that  $I(X) \in \mathcal{I}$ . For each morphism  $f : X \rightarrow Y$ , we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & I(X) & \xrightarrow{p_X} & \Sigma X \longrightarrow 0 \\ & & \downarrow f & & \downarrow I(f) & & \downarrow \Sigma f \\ 0 & \longrightarrow & Y & \xrightarrow{i_Y} & I(Y) & \xrightarrow{p_Y} & \Sigma Y \longrightarrow 0 \end{array}$$

with rows in  $\mathcal{S}$ . It is easily seen that  $\underline{\Sigma f}$  does not depend on the choice of  $I(f)$ . We define  $\underline{\Sigma f} = \underline{\Sigma f}$ .

**Lemma 4.1.** *Let*

$$0 \rightarrow X_1 \oplus I_1 \xrightarrow{\begin{pmatrix} f_{11} & f_{12} \\ f_{13} & f_{14} \end{pmatrix}} X_2 \oplus I_2 \xrightarrow{\begin{pmatrix} f_{21} & f_{22} \end{pmatrix}} X_3 \rightarrow 0$$

be a short exact sequence in  $\mathcal{S}$ , where  $I_1, I_2 \in \mathcal{I}$ . Then the sequence

$$\underline{\mathcal{B}}(-, X_1) \xrightarrow{\underline{\mathcal{B}}(-, f_{11})} \underline{\mathcal{B}}(-, X_2) \xrightarrow{\underline{\mathcal{B}}(-, f_{21})} \underline{\mathcal{B}}(-, X_3)$$

is exact.

*Proof.* Since  $f_{21}f_{11} + f_{22}f_{13} = 0$ , we have  $f_{21}f_{11} = 0$ . Consequently,  $\text{Im } \underline{\mathcal{B}}(-, f_{11}) \subseteq \ker \underline{\mathcal{B}}(-, f_{21})$ . Assume that  $\underline{g} : Y \rightarrow X_2$  is a morphism in  $\ker \underline{\mathcal{B}}(Y, f_{21})$ , then we have  $\underline{f_{21}g} = 0$ . There exist two morphisms  $a : Y \rightarrow I$  and  $b : I \rightarrow X_3$  for some  $I \in \mathcal{I}$  such that  $f_{21}g = ba$ . Since  $I$  is projective-injective, we can assume that  $b = (f_{21}, f_{22}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ . We note that  $(f_{21}, f_{22}) \begin{pmatrix} g - x_1 a \\ -x_2 a \end{pmatrix} = 0$ . There exists a morphism  $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : Y \rightarrow X_1 \oplus I_1$  such that  $\begin{pmatrix} g - x_1 a \\ -x_2 a \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{13} & f_{14} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ . Thus  $\underline{g} = \underline{h_1 f_{11}} \in \text{Im } \underline{\mathcal{B}}(Y, f_{11})$ . Therefore,  $\ker \underline{\mathcal{B}}(Y, f_{21}) = \text{Im } \underline{\mathcal{B}}(Y, f_{11})$ .  $\square$

**Lemma 4.2.** *Assume that the following diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{i_1} & I(X_1) & \xrightarrow{p_1} & \Sigma X_1 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow a_1 & & \parallel \\ 0 & \longrightarrow & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \Sigma X_1 \longrightarrow 0 \end{array} \quad (4.1)$$

is commutative with rows in  $\mathcal{S}$ . Then the complex

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1 \quad (4.2)$$

belongs to  $C_{3-\Sigma}^{\text{ex}}(\iota(\underline{\mathcal{B}}))$ .

*Proof.* We check the lemma without using Lemma 2.2. By the definition of  $\Sigma$ , we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{i_1} & I(X_1) & \xrightarrow{p_1} & \Sigma X_1 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow I(f_1) & & \downarrow \Sigma f_1 \\ 0 & \longrightarrow & X_2 & \xrightarrow{i_2} & I(X_2) & \xrightarrow{p_2} & \Sigma X_2 \longrightarrow 0 \end{array}$$

with rows in  $\mathcal{S}$ . Since the left square in diagram (4.1) is a pushout, there exists a unique morphism  $a_2 : X_3 \rightarrow I(X_2)$  such that  $i_2 = a_2 f_2$  and  $I(f_1) = a_2 a_1$ . Noting that

$$(p_2 a_2) f_2 = p_2 i_2 = 0 = (\Sigma f_1 \cdot f_3) f_2,$$

$$(p_2 a_2) a_1 = p_2 I(f_1) = \Sigma f_1 \cdot p_1 = (\Sigma f_1 \cdot f_3) a_1,$$

we obtain  $p_2 a_2 = \Sigma f_1 \cdot f_3$  by the universal property of pushout. Thus we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{\begin{pmatrix} f_1 \\ i_1 \end{pmatrix}} & X_2 \oplus I(X_1) & \xrightarrow{(f_2, -a_1)} & X_3 \longrightarrow 0 \\ & & \parallel & & \downarrow (0,1) & & \downarrow -f_3 \\ 0 & \longrightarrow & X_1 & \xrightarrow{i_1} & I(X_1) & \xrightarrow{p_1} & \Sigma X_1 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow a_1 & & \parallel \\ 0 & \longrightarrow & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \Sigma X_1 \longrightarrow 0 \\ & & \parallel & & \downarrow a_2 & & \downarrow \Sigma f_1 \\ 0 & \longrightarrow & X_2 & \xrightarrow{i_2} & I(X_2) & \xrightarrow{p_2} & \Sigma X_2 \longrightarrow 0 \end{array}$$

with rows in  $\mathcal{S}$ .

Therefore, we have three short exact sequences as follows:

$$0 \rightarrow X_1 \xrightarrow{\begin{pmatrix} f_1 \\ i_1 \end{pmatrix}} X_2 \oplus I(X_1) \xrightarrow{(f_2, -a_1)} X_3 \rightarrow 0,$$

$$0 \rightarrow X_2 \oplus I(X_1) \xrightarrow{\begin{pmatrix} f_2 & -a_1 \\ 0 & 1 \end{pmatrix}} X_3 \oplus I(X_1) \xrightarrow{(f_3, p_1)} \Sigma X_1 \rightarrow 0,$$

$$0 \rightarrow X_3 \xrightarrow{\begin{pmatrix} f_3 \\ a_2 \end{pmatrix}} \Sigma X_1 \oplus I(X_2) \xrightarrow{(\Sigma f_1, -p_2)} \Sigma X_2 \rightarrow 0.$$

Now the lemma follows from Lemma 4.1.  $\square$

Denote by  $\Delta$  the collection of 3- $\Sigma$ -sequences in  $C_{3-\Sigma}^{\text{ex}}(\iota(\underline{\mathcal{B}}))$  which are isomorphic to the ones in the form of (4.2).

**Proposition 4.3.** *The functor  $Z_1 : \Delta \rightarrow \text{mod } \underline{\mathcal{B}}$  is dense.*

*Proof.* We check the result without using Theorem 3.9. For each  $M \in \text{mod } \mathcal{C}$ , we choose a projective presentation

$$\underline{\mathcal{B}}(-, X_1) \xrightarrow{\underline{\mathcal{B}}(-, f_1)} \underline{\mathcal{B}}(-, X_2) \rightarrow M \rightarrow 0.$$

Then we have a commutative diagram (4.1). By Lemma 4.2, we get

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \Sigma X_1) \in \Delta.$$

Note that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_2 & \xrightarrow{i_2} & I(X_2) & \xrightarrow{p_2} & \Sigma X_2 \longrightarrow 0 \\ & & \downarrow f_2 & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \parallel \\ 0 & \longrightarrow & X_3 & \xrightarrow{\begin{pmatrix} f_3 \\ a_2 \end{pmatrix}} & \Sigma X_1 \oplus I(X_2) & \xrightarrow{(-\Sigma f_1, p_2)} & \Sigma X_2 \longrightarrow 0 \end{array}$$

with rows in  $\mathcal{S}$ . Lemma 4.2 implies that  $X_\bullet[1] \in \Delta$ . Therefore,  $X_\bullet[2] \in \Delta$  and  $Z_1(X_\bullet[2]) = M$ .  $\square$

**Example 4.4.** ([13, Theorem 2.6]) Let  $(\mathcal{B}, \mathcal{S})$  be a Frobenius category, then  $(\underline{\mathcal{B}}, \Sigma, \Delta)$  is a triangulated category.

**4.2. Standard construction of  $n$ -angulated categories.** In this subsection, we assume that  $\mathcal{T}$  is a triangulated category with the suspension functor  $\Sigma$  and  $\mathcal{C}$  is a full subcategory of  $\mathcal{T}$ . Suppose  $d$  is a positive integer. Recall that the subcategory  $\mathcal{C}$  is called  *$d$ -rigid* if  $\mathcal{T}(\mathcal{C}, \Sigma^i \mathcal{C}) = 0$  for  $i = 1, 2, \dots, d-1$ . The subcategory  $\mathcal{C}$  is called  *$d$ -cluster tilting* if  $\mathcal{C}$  is functorially finite and

$$\begin{aligned} \mathcal{C} &= \{X \in \mathcal{T} \mid \mathcal{T}(X, \Sigma^i \mathcal{C}) = 0, \forall i = 1, \dots, d-1\} \\ &= \{X \in \mathcal{T} \mid \mathcal{T}(\mathcal{C}, \Sigma^i X) = 0, \forall i = 1, \dots, d-1\}. \end{aligned}$$

Let  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{Z}$  be full subcategories of  $\mathcal{T}$ . We denote by  $\mathcal{X} * \mathcal{Y}$  the class of objects  $T$  in  $\mathcal{T}$  such that there is a triangle  $X \rightarrow T \rightarrow Y \rightarrow \Sigma X$  with  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ . The octahedral axiom implies that  $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$ .

**Remark 4.5.** (a) Let  $\mathcal{C}$  be a  $d$ -rigid subcategory which is closed under  $\Sigma^d$ , then  $\mathcal{T}(\mathcal{C}, \Sigma^i \mathcal{C}) = 0$  for each  $i \notin d\mathbb{Z}$ .

(b) The subcategory  $\mathcal{C}$  is  $d$ -cluster tilting if and only if  $\mathcal{C}$  is  $d$ -rigid and  $\mathcal{T} = \mathcal{C} * \Sigma \mathcal{C} * \Sigma^2 \mathcal{C} * \dots * \Sigma^{d-1} \mathcal{C}$ .

*Proof.* (a) It is clear. (b) It is proved in [25].  $\square$

**Lemma 4.6.** Let  $\mathcal{C}$  be a  $d$ -rigid subcategory which is closed under  $\Sigma^d$ . Suppose that there are the following triangles:

$$\begin{aligned} X_1 &\xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha'_2} X_{2.5} \xrightarrow{\alpha_{d+2}^1} \Sigma X_1, \\ X_{i.5} &\xrightarrow{\alpha''_i} X_{i+1} \xrightarrow{\alpha'_{i+1}} X_{i+1.5} \xrightarrow{\alpha_{d+2}^i} \Sigma X_{i.5}, \\ X_{d.5} &\xrightarrow{\alpha''_d} X_{d+1} \xrightarrow{\alpha_{d+1}} X_{d+2} \xrightarrow{\alpha_{d+2}^d} \Sigma X_{d.5}, \end{aligned}$$

where  $2 \leq i \leq d-1$ ,  $X_j \in \mathcal{C}$  for  $j \in \mathbb{Z}$  and  $X_k \in \mathcal{T}$  for  $k \notin \mathbb{Z}$ . Then

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{d+1}} X_{d+2} \xrightarrow{\alpha_{d+2}} \Sigma^d X_1 \quad (4.3)$$

belongs to  $C_{(d+2)-\Sigma^d}^{ex}(\iota(\mathcal{C}))$ , where  $\alpha_i = \alpha''_i \alpha'_i$  for  $2 \leq i \leq d$  and

$$\alpha_{d+2} = \Sigma^{d-1} \alpha_{d+2}^1 \cdot \Sigma^{d-2} \alpha_{d+2}^2 \cdot \dots \cdot \Sigma \alpha_{d+2}^{d-1} \cdot \alpha_{d+2}^d.$$

*Proof.* For each object  $A \in \mathcal{C}$ , applying the functor  $\mathcal{T}(A, -)$  to the above triangles, we obtain the following exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{T}(A, \Sigma^{-1}X_{2.5}) \rightarrow \mathcal{T}(A, X_1) \rightarrow \mathcal{T}(A, X_2) \rightarrow \mathcal{T}(A, X_{2.5}) \rightarrow 0, \\ 0 &\rightarrow \mathcal{T}(A, \Sigma^{-1}X_{i+1.5}) \rightarrow \mathcal{T}(A, X_{i.5}) \rightarrow \mathcal{T}(A, X_{i+1}) \rightarrow \mathcal{T}(A, X_{i+1.5}) \rightarrow \mathcal{T}(A, \Sigma X_{i.5}) \rightarrow 0, \\ 0 &\rightarrow \mathcal{T}(A, X_{d.5}) \rightarrow \mathcal{T}(A, X_{d+1}) \rightarrow \mathcal{T}(A, X_{d+2}) \rightarrow \mathcal{T}(A, \Sigma X_{d.5}) \rightarrow 0, \end{aligned}$$

where  $2 \leq i \leq d-1$ . Moreover, for  $1 \leq i \leq d-2$  and  $2 \leq j \leq d-1$ , we have  $\mathcal{T}(A, \Sigma^i X_{2.5}) = 0$ ,  $\mathcal{T}(A, \Sigma^j X_{d.5}) = 0$  and  $\mathcal{T}(A, \Sigma^i \alpha_{d+2}^j) : \mathcal{T}(A, \Sigma^i X_{j+1.5}) \rightarrow \mathcal{T}(A, \Sigma^{i+1} X_{j.5})$  is an isomorphism. Since  $\mathcal{T}(A, \Sigma X_{i.5}) \cong \mathcal{T}(A, \Sigma^{i-1} X_{2.5}) = 0$  and  $\mathcal{T}(A, \Sigma^{-1} X_{i+1.5}) \cong \mathcal{T}(\Sigma^d A, \Sigma^{d-1} X_{i+1.5}) \cong \mathcal{T}(\Sigma^d A, \Sigma^i X_{d.5}) = 0$  for  $2 \leq i \leq d-1$ , gluing the above exact sequences, we have the following long exact sequence

$$\mathcal{T}(A, X_1) \rightarrow \mathcal{T}(A, X_2) \rightarrow \cdots \rightarrow \mathcal{T}(A, X_{d+2}) \rightarrow \mathcal{T}(A, \Sigma^d X_1) \rightarrow \mathcal{T}(A, \Sigma^d X_2),$$

where the last but one morphism is the composition

$$\mathcal{T}(A, X_{d+2}) \twoheadrightarrow \mathcal{T}(A, \Sigma X_{d.5}) \xrightarrow{\simeq} \mathcal{T}(A, \Sigma^2 X_{d-1.5}) \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \mathcal{T}(A, \Sigma^{d-1} X_{2.5}) \hookrightarrow \mathcal{T}(A, \Sigma^d X_1).$$

□

Denote by  $\Theta$  the class of  $(d+2)$ - $\Sigma^d$ -sequences in  $C_{(d+2)-\Sigma^d}^{\text{ex}}(\mathcal{U}(\mathcal{C}))$  of the form (4.3).

**Proposition 4.7.** *Let  $\mathcal{C}$  be a  $d$ -cluster tilting subcategory which is closed under  $\Sigma^d$ , then the functor  $Z_1 : \Theta \rightarrow \text{mod } \mathcal{C}$  is dense.*

*Proof.* We prove the proposition without using Theorem 3.9. For each  $M \in \text{mod } \mathcal{C}$ , there exists an exact sequence  $\mathcal{C}(-, X_1) \xrightarrow{\mathcal{C}(-, f_1)} \mathcal{C}(-, X_2) \rightarrow M \rightarrow 0$ . Assume that  $X_1 \xrightarrow{f_1} X_2 \rightarrow X_{2.5} \rightarrow \Sigma X_1$  is a triangle. Then it is easy to see that  $M \cong \mathcal{T}(-, X_{2.5})|_{\mathcal{C}}$ . It follows from Remark 4.5 that  $X_{2.5} \in \mathcal{T} = \mathcal{C} * \Sigma \mathcal{C} * \Sigma^2 \mathcal{C} * \cdots * \Sigma^{d-1} \mathcal{C}$ . Thus we have the following triangles:

$$\begin{aligned} X_{i.5} &\rightarrow X_{i+1} \rightarrow X_{i+1.5} \rightarrow \Sigma X_{i.5}, \\ X_{d.5} &\rightarrow X_{d+1} \rightarrow X_{d+2} \rightarrow \Sigma X_{d+1.5}, \end{aligned}$$

where  $2 \leq i \leq d-1$ . By Lemma 4.6, we have a complex  $X_{\bullet} = (X_1 \xrightarrow{f_1} X_2 \rightarrow \cdots \rightarrow X_{d+2} \rightarrow \Sigma^d X_1) \in \Theta$ . It is easy to see that  $T_M = X_{\bullet}[2] \in \Theta$  and  $Z_1 T_M = M$ . □

**Example 4.8.** ([11, Theorem 1]) Let  $\mathcal{T}$  be a triangulated category with the suspension functor  $\Sigma$  and  $\mathcal{C}$  be a  $d$ -cluster tilting subcategory such that  $\Sigma^d \mathcal{C} \subseteq \mathcal{C}$ , then  $(\mathcal{C}, \Sigma^d, \Theta)$  is a  $(d+2)$ -angulated category.

**4.3.  $n$ -angulated categories from local rings.** Let  $R$  be a commutative local ring with a principle maximal ideal  $\mathfrak{m} = (p)$  such that  $\mathfrak{m}^2 = 0$ . Then  $\mathfrak{m}$  is the unique nontrivial ideal of  $R$ . By the Baer Criterion, it is easy to check that  $R$  is a selfinjective ring. Moreover, we have  $\text{mod } R = \text{add}(R \oplus \mathfrak{m})$ , where  $\text{mod } R$  is the category of finitely generated  $R$ -modules. We denote by  $\mathcal{C}$  the category of finitely generated projective  $R$ -modules and by  $\Sigma$  the identity functor of  $\mathcal{C}$ .

**Lemma 4.9.** *Each minimal  $n$ - $\Sigma$ -periodic injective resolution of  $\mathfrak{m}$  is in the form of*

$$R(u)_{\bullet} = (R \xrightarrow{up} R \xrightarrow{p} \cdots \xrightarrow{p} R \xrightarrow{p} \Sigma R)$$

for some unit  $u$  in  $R$ . Moreover,  $R(u)_{\bullet}$  is  $n$ - $\Sigma$ -homotopy equivalent to  $R(v)_{\bullet}$  if and only if  $up = vp$ .

*Proof.* It is clear that  $R(u)_\bullet$  is a minimal  $n$ - $\Sigma$ -periodic injective resolution of  $\mathfrak{m}$  for each unit  $u$  in  $R$ . Conversely, let  $T_{\mathfrak{m}}$  be a minimal  $n$ - $\Sigma$ -periodic injective resolution of  $\mathfrak{m}$ . We can assume that

$$T_{\mathfrak{m}} = (R \xrightarrow{u_1 p} R \xrightarrow{u_2 p} \cdots \xrightarrow{u_{n-1} p} R \xrightarrow{u_n p} \Sigma R)$$

where all  $u_i$  are units in  $R$ . It is obvious that  $T_{\mathfrak{m}}$  is isomorphic to

$$R(u)_\bullet = (R \xrightarrow{up} R \xrightarrow{p} \cdots \xrightarrow{p} R \xrightarrow{p} \Sigma R)$$

where  $u = u_1 u_2 \cdots u_n$ . Assume that  $R(u)_\bullet$  is  $n$ - $\Sigma$ -homotopy equivalent to  $R(v)_\bullet$ , then we have the following commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{up} & R & \xrightarrow{p} & \cdots & \xrightarrow{p} & R \xrightarrow{p} \Sigma R \\ \downarrow w_1 & & \downarrow w_2 & & & & \downarrow w_n \quad \downarrow \Sigma w_1 \\ R & \xrightarrow{vp} & R & \xrightarrow{p} & \cdots & \xrightarrow{p} & R \xrightarrow{p} \Sigma R \\ \downarrow w'_1 & & \downarrow w'_2 & & & & \downarrow w'_n \quad \downarrow \Sigma w'_1 \\ R & \xrightarrow{up} & R & \xrightarrow{p} & \cdots & \xrightarrow{p} & R \xrightarrow{p} \Sigma R. \end{array}$$

Moreover, there exist morphisms  $h_i : R \rightarrow R$  for  $1 \leq i \leq n$  such that  $w'_1 w_1 - 1 = h_1 up + ph_n$  and  $w'_2 w_2 - 1 = h_2 p + uph_1$ . Noting that  $p = w'_1 w_1 p = w'_2 w_2 p$ ,  $uw'_1 p = vw'_2 p$  and  $w_2 p = w_3 p = \cdots = w_n p = w_1 p$ , we have  $up = uw'_1 w_1 p = vw'_2 w_1 p = vw'_2 w_2 p = vp$ . The “if” part is trivial.  $\square$

For each unit  $u$  in  $R$ , we denote by  $\Theta_u$  the class of  $n$ - $\Sigma$ -sequences in  $C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  consisting of  $F_\bullet \oplus A_\bullet$ , where  $F_\bullet$  is a finite direct sum of  $R(u)_\bullet$  and  $A_\bullet$  is a contractible  $n$ - $\Sigma$ -sequence.

**Proposition 4.10.** *The functor  $Z_1 : \Theta_u \rightarrow \text{mod } R$  is dense.*

*Proof.* Since  $\text{mod } R = \text{add}(R \oplus \mathfrak{m})$ ,  $Z_1 R(u)_\bullet = \mathfrak{m}$  and  $Z_1 A_\bullet = R$ , where  $A_\bullet = (R \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma R \xrightarrow{1} \Sigma R)$ , the functor  $Z_1$  is dense.  $\square$

**Example 4.11.** ([6, Theorem 3.6]) For each unit  $u$  in  $R$ , the triple  $(\mathcal{C}, \Sigma, \Theta_u)$  is an  $n$ -angulated category whenever  $n$  is even, or when  $n$  is odd and  $2p = 0$ .

**Remark 4.12.** If  $n$  is odd and  $2p \neq 0$ , then by Lemma 4.9,  $R(-u)_\bullet \notin \Theta_u$  for each unit  $u$  in  $R$ . But  $R(u)_\bullet[1] \cong R(-u)_\bullet$ , thus  $(\mathcal{C}, \Sigma, \Theta_u)$  is not an  $n$ -angulated category since  $\Theta_u$  is not closed under the translation functor. In fact,  $(\mathcal{C}, \Sigma)$  does not admit any  $n$ -angulation in this case.

## 5. MORE EXAMPLES

In this section, we will first discuss the  $n$ -angulated structure of semisimple categories, then apply Corollary 3.11 to get a class of new examples of  $n$ -angulated categories from quasi-periodic selfinjective algebras. At last, we construct other  $n$ -angulations from known ones.



**5.1. Semisimple categories.** We recall that an additive category  $\mathcal{C}$  is *semisimple* if each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  factors as  $f = hg$ , where  $g$  is a split epimorphism and  $h$  is a split monomorphism.

**Remark 5.1.** Let  $\mathcal{C}$  be a semisimple category. Then  $\mathcal{C}$  is idempotent complete. Moreover, for each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \alpha & & \downarrow \beta \\ A \oplus B & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} & C \oplus B \end{array}$$

where  $\alpha$  and  $\beta$  are isomorphisms.

We recall the following description of semisimple categories given by Jasso.

**Lemma 5.2.** ([17, Theorem 3.9]) *Let  $m$  be a positive integer. Then the  $m$ -abelian categories in which every  $m$ -exact sequence is contractible are precisely the semisimple categories.*

We can compare Lemma 5.2 with the following.

**Theorem 5.3.** *Let  $\mathcal{C}$  be an additive category and  $\Sigma$  an automorphism of  $\mathcal{C}$ . Then  $(\mathcal{C}, \Sigma)$  is an  $n$ -angulated category in which each  $n$ -angle is contractible if and only if  $\mathcal{C}$  is semisimple.*

*Proof.* Assume that  $\mathcal{C}$  is semisimple. Denote by  $\Theta$  the class of all contractible  $n$ - $\Sigma$ -sequences. We claim that  $(\mathcal{C}, \Sigma, \Theta)$  is an  $n$ -angulated category. Indeed, it is easily seen that  $\Theta$  is closed under isomorphisms, direct sums, direct summands and rotations. For each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , by Remark 5.1 we can assume that  $f$  is the morphism of the form  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : A \oplus B \rightarrow C \oplus B$ . Thus there exists the following contractible  $n$ - $\Sigma$ -sequence

$$A \oplus B \xrightarrow{f} C \oplus B \xrightarrow{(1,0)} C \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Sigma A \oplus \Sigma B.$$

Since  $\mathcal{C}$  is idempotent complete, each contractible  $n$ - $\Sigma$ -sequence is a direct sum of trivial  $n$ -angles. It is easy to see that (N3) holds. Let  $\varphi_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$  be a morphism of contractible  $n$ - $\Sigma$ -sequences. Since the mapping cone  $C(\varphi_{\bullet})$  is isomorphic to  $X_{\bullet}[1] \oplus Y_{\bullet}$ , axiom (N4) holds.

Conversely, assume that  $\mathcal{C}$  is an  $n$ -angulated category in which each  $n$ -angle is contractible. Given a morphism  $f_1 : X_1 \rightarrow X_2$  in  $\mathcal{C}$ , suppose that  $X_{\bullet} = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$  is an  $n$ -angle. Since  $X_{\bullet} \in C_{n-\Sigma}^{\text{ex}}(\iota(\mathcal{C}))$  is contractible,  $\mathcal{C}(-, f_1)$  has a factorization

$$\begin{array}{ccc} \mathcal{C}(-, X_1) & \xrightarrow{\mathcal{C}(-, f_1)} & \mathcal{C}(-, X_2) \\ & \searrow a_1 & \nearrow b_1 \\ & M & \end{array}$$

for some  $M \in \text{mod } \mathcal{C}$  such that  $a_1$  is a split epimorphism and  $b_1$  is a split monomorphism (see [11, Section 2.3]). It follows from Theorem 3.9 that  $\text{mod } \mathcal{C} = \text{proj } \mathcal{C}$ . So we can assume that  $M = \mathcal{C}(-, Y)$ ,  $a_1 = \mathcal{C}(-, f'_1)$  and  $b_1 = \mathcal{C}(-, f''_1)$ . The Yoneda Lemma shows that  $f_1 = f''_1 f'_1$  where  $f'_1$  is a split epimorphism and  $f''_1$  is a split monomorphism. Therefore,  $\mathcal{C}$  is a semisimple category by definition.  $\square$

The following result characterizes semisimple categories in terms of  $n$ -angulated categories and  $m$ -abelian categories.

**Corollary 5.4.** *Let  $\mathcal{C}$  be an additive category,  $m$  and  $n$  be positive integers where  $n \geq 3$ . Then  $\mathcal{C}$  is an  $n$ -angulated and  $m$ -abelian category if and only if  $\mathcal{C}$  is a semisimple category.*

*Proof.* The “if” part follows from Theorem 5.3 and Lemma 5.2. For the “only if” part, we take an  $n$ -exact sequence  $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$ . Since  $f_0$  is a monomorphism, we infer that  $f_0$  is a split monomorphism by Lemma 2.4. Thus all the  $n$ -exact sequences are contractible by [17, Proposition 2.6]. Lemma 5.2 implies that  $\mathcal{C}$  is a semisimple category.  $\square$

**5.2. Quasi-periodic selfinjective algebras.** Let  $k$  be a field and  $A$  be a finite-dimensional  $k$ -algebra. Given an algebra automorphism  $\sigma$  of  $A$ , we denote by  ${}_1A_\sigma$  the bimodule structure on  $A$  where the right action is twisted by  $\sigma$ . It is easy to check that, given two automorphisms  $\sigma$  and  $\tau$ , there is an isomorphism  ${}_1A_\sigma \otimes_A ({}_1A_\tau) \cong {}_1A_{\tau\sigma}$ . A finite-dimensional  $k$ -algebra  $A$  is said to be *quasi-periodic* if  $A$  has a quasi-periodic projective resolution over the enveloping algebra  $A^e = A^{\text{op}} \otimes_k A$ , i.e.,  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$  as  $A$ - $A$ -bimodule for some natural number  $n$  and some automorphism  $\sigma$  of  $A$ . In particular,  $A$  is *periodic* if  $\Omega_{A^e}^n(A) \cong A$ .

In this subsection, we assume that  $A$  is a finite-dimensional indecomposable  $k$ -algebra such that  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$  as an  $A$ - $A$ -bimodule for some automorphism  $\sigma$  of  $A$  and for some  $n \geq 3$ . Since an indecomposable quasi-periodic algebra  $A$  is self-injective by [12, Lemma 1.5], the category  $\text{mod } A$  is a Frobenius category. Denote by  $\text{proj } A$  the category of finitely generated projective modules over  $A$ . Note that  $e_i A \otimes_A ({}_1A_{\sigma^{-1}}) \cong e_i A_{\sigma^{-1}} \cong \sigma(e_i)A$  for each idempotent  $e_i$  of  $A$ , hence the functor  $-\otimes_A ({}_1A_{\sigma^{-1}}) : \text{proj } A \rightarrow \text{proj } A$  is an automorphism.

**Theorem 5.5.** *The category  $(\text{proj } A, -\otimes_A ({}_1A_{\sigma^{-1}}))$  admits an  $n$ -angulation.*

*Proof.* Since  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ , there exists an exact sequence of  $A$ - $A$ -bimodules

$$0 \rightarrow {}_1A_\sigma \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow A \rightarrow 0 \quad (5.1)$$

where the  $P_i$ 's are projective as bimodules. Tensoring this sequence with  ${}_1A_{\sigma^{-1}}$  yields the following exact sequence of  $A$ - $A$ -bimodules

$$0 \rightarrow A \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_n \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_{n-1} \rightarrow \dots \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_1 \rightarrow {}_1A_{\sigma^{-1}} \rightarrow 0$$

where all the  ${}_1A_{\sigma^{-1}} \otimes_A P_i$  are projective as bimodules. Thus we have the following exact sequence of exact endofunctors of  $\text{mod } A$

$$\begin{aligned} 0 \rightarrow \text{Id} \rightarrow -\otimes_A ({}_1A_{\sigma^{-1}}) \otimes_A P_n \rightarrow -\otimes_A ({}_1A_{\sigma^{-1}}) \otimes_A P_{n-1} \rightarrow \dots \\ \dots \rightarrow -\otimes_A ({}_1A_{\sigma^{-1}}) \otimes_A P_1 \rightarrow -\otimes_A ({}_1A_{\sigma^{-1}}) \rightarrow 0. \end{aligned}$$

Moreover, the functors  $-\otimes_A ({}_1A_{\sigma^{-1}}) \otimes_A P_i$  take values in  $\text{proj } A$ . By Corollary 3.11,  $(\text{proj } A, -\otimes_A ({}_1A_{\sigma^{-1}}))$  admits an  $n$ -angulation.  $\square$

**Corollary 5.6.** *For each positive integer  $m$ , the category  $(\text{proj } A, -\otimes_A ({}_1A_{\sigma^{-m}}))$  admits an  $mn$ -angulation. In particular, if  $\sigma$  is of finite order  $l$ , then  $(\text{proj } A, \text{Id}_{\text{proj } A})$  admits an  $ln$ -angulation.*

*Proof.* Since  $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ , we claim that  $\Omega_{A^e}^{mn}(A) \cong {}_1A_{\sigma^m}$ , thus the corollary follows from Theorem 5.5. Indeed, we only need to show that there exists an exact sequence of  $A$ - $A$ -bimodules

$$0 \rightarrow {}_1A_{\sigma^m} \rightarrow P_{mn} \rightarrow P_{mn-1} \rightarrow \cdots \rightarrow P_{(m-1)n+1} \rightarrow \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow A \rightarrow 0$$

where the  $P_i$ 's are projective as bimodules. We will prove it by induction on  $m$ . It is trivial for  $m = 1$ . Assume that  $m > 1$  and our claim holds for  $m - 1$ . Applying the functor  ${}_1A_{\sigma^{m-1}} \otimes_A -$  to the sequence (5.1), we obtain the following exact sequence of  $A$ - $A$ -bimodules

$$0 \rightarrow {}_1A_{\sigma^m} \rightarrow {}_1A_{\sigma^{m-1}} \otimes_A P_n \rightarrow \cdots \rightarrow {}_1A_{\sigma^{m-1}} \otimes_A P_1 \rightarrow {}_1A_{\sigma^{m-1}} \rightarrow 0. \quad (5.2)$$

We take  $P_{(m-1)n+i} = {}_1A_{\sigma^{m-1}} \otimes_A P_i$ , where  $i = 1, 2, \dots, n$ . Then the  $P_{(m-1)n+i}$ 's are projective as bimodules. By induction and the sequence (5.2), our claim holds for each positive integer  $m$ .  $\square$

**5.3. Global automorphisms.** Given an  $n$ -angulation  $\Theta$ , we want to construct more  $n$ -angulations from known ones. For this, global automorphisms of  $(\mathcal{C}, \Sigma)$ , which were introduced by Balmer, play an important role.

**Definition 5.7.** ([3]) Let  $\mathcal{C}$  be an additive category and  $\Sigma$  be an automorphism of  $\mathcal{C}$ . A *global automorphism*  $\alpha$  of  $(\mathcal{C}, \Sigma)$  is an invertible endomorphism of the identity functor  $Id : \mathcal{C} \rightarrow \mathcal{C}$  which commutes with  $\Sigma$ . In other words, a global automorphism  $\alpha$  is a collection of isomorphisms  $\alpha_A : A \rightarrow A$ , for all objects  $A$  in  $\mathcal{C}$ , such that  $\alpha_B f = f \alpha_A$  for each morphism  $f : A \rightarrow B$  and  $\alpha_{\Sigma A} = \Sigma \alpha_A$  for each object  $A$  in  $\mathcal{C}$ .

**Example 5.8.** (a) Let  $k$  be a field,  $\mathcal{C}$  be a  $k$ -linear category and  $\Sigma$  be a  $k$ -linear automorphism of  $\mathcal{C}$ . Then for each  $\lambda \in k \setminus \{0\}$ ,  $\alpha = \{\alpha_A = \lambda 1_A : A \rightarrow A | A \in \mathcal{C}\}$  is a global automorphism of  $(\mathcal{C}, \Sigma)$ .

(b) Let  $R$  be a commutative ring,  $\mathcal{C}$  be the category of finitely generated projective modules over  $R$  and  $\Sigma$  be the identity functor of  $\mathcal{C}$ . Then for each unit  $u$  in  $R$ , multiplication by  $u$  gives a global automorphism  $\lambda_u$  of  $(\mathcal{C}, \Sigma)$ .

The following is a higher version of [3, Proposition 4].

**Proposition 5.9.** *Let  $\Theta$  be an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$  and  $\alpha$  be a global automorphism of  $(\mathcal{C}, \Sigma)$ . Define  $\Theta^\alpha$  as the class of  $n$ - $\Sigma$ -sequences*

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1)$$

such that

$$X_\bullet^\alpha = (X_1 \xrightarrow{f_1 \alpha_{X_1}} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1) \in \Theta.$$

Then  $\Theta^\alpha$  is an  $n$ -angulation of  $(\mathcal{C}, \Sigma)$ .

*Proof.* For each  $M \in \text{mod } \mathcal{C}$ ,  $X_\bullet$  is an  $n$ - $\Sigma$ -periodic injective resolution of  $M$  if and only if so is  $X_\bullet^\alpha$  since  $\alpha_{X_1}$  is an isomorphism. By Theorem 3.9, we only need to show that  $\Theta^\alpha$  satisfies (N4).

Given a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in  $\Theta^\alpha$ , we have the following commutative diagram

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_1 \alpha_{X_1}} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\
 \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_n \\
 Y_1 & \xrightarrow{g_1 \alpha_{Y_1}} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma Y_1
 \end{array}$$

with rows in  $\Theta$ , which can be completed to a whole commutative diagram such that the mapping cone  $C(\varphi_\bullet)$  belongs to  $\Theta$  by (N4). The following commutative diagram

$$\begin{array}{ccccccc}
 X_2 \oplus Y_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \alpha_{Y_1} \end{pmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} & \cdots & \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} & \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma(f_1 \alpha_{X_1}) & 0 \\ \Sigma \varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1 \\
 \downarrow \begin{pmatrix} \alpha_{X_2}^{-1} & 0 \\ 0 & 1 \end{pmatrix} & & \parallel & & & & \parallel \\
 X_2 \oplus Y_1 & \xrightarrow{\begin{pmatrix} -f_2 \alpha_{X_2} & 0 \\ \varphi_2 \alpha_{X_2} & g_1 \alpha_{Y_1} \end{pmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} & \cdots & \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} & \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ \Sigma \varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1 \\
 & & & & & & \downarrow \begin{pmatrix} \Sigma \alpha_{X_2}^{-1} & 0 \\ 0 & 1 \end{pmatrix}
 \end{array}$$

implies that the second row belongs to  $\Theta$  since the first row  $C(\varphi_\bullet) \in \Theta$ . Noting that

$$\begin{pmatrix} -f_2 \alpha_{X_2} & 0 \\ \varphi_2 \alpha_{X_2} & g_1 \alpha_{Y_1} \end{pmatrix} = \begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix} \begin{pmatrix} \alpha_{X_2} & 0 \\ 0 & \alpha_{Y_1} \end{pmatrix} = \begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix} \alpha_{X_2 \oplus Y_1},$$

we obtain that

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ \Sigma \varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to  $\Theta^\alpha$ .  $\square$

**Remark 5.10.** We denote by  $\text{G-Aut}(\mathcal{C}, \Sigma)$  the set of all global automorphisms of  $(\mathcal{C}, \Sigma)$ . It is easy to see that  $\text{G-Aut}(\mathcal{C}, \Sigma)$  is a group under composition of natural transformations. Proposition 5.9 implies that the group  $\text{G-Aut}(\mathcal{C}, \Sigma)$  acts on the set of  $n$ -angulations of  $(\mathcal{C}, \Sigma)$  from the right. In general, the action is not free, but we do not know whether the action is transitive or not.

**Example 5.11.** In Example 4.11, for each  $n$ -angulation  $\Theta$  of  $(\mathcal{C}, \Sigma)$ , by Lemma 4.9, it is not hard to show that  $\Theta = \Theta_u$  for some unit  $u$  in  $R$ . Moreover,  $\Theta_u = \Theta_v$  if and only if  $up = vp$ . By Example 5.8(b) and Proposition 5.9, we have  $\Theta_u = \Theta_1^{\lambda_u}$ , where 1 is the identity element of  $R$  and  $\lambda_u$  is the global automorphism induced by  $u$ . Thus in this case the group of global automorphism of  $(\mathcal{C}, \Sigma)$  acts transitively on the set of all  $n$ -angulations. But the action is not free.

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